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SOCIALLY STRUCTURED GAMES

ABSTRACT. We generalize the concept of a cooperative non-transferable utility game by introducing a socially structured game. In a socially structured game every coalition of players can organize themselves according to one or more internal organizations to generate payoffs. Each admissible internal organization on a coalition yields a set of payoffs attainable by the members of this coalition. The strengths of the players within an internal organization depend on the structure of the internal organization and are represented by an exogenously given power vector. More powerful players have the power to take away payoffs of the less powerful players as long as those latter players are not able to guarantee their payoffs by forming a different internal organization within some coalition in which they have more power.

We introduce the socially stable core as a solution concept that contains those payoffs that are both stable in an economic sense, i.e., belong to the core of the underlying cooperative game, and stable in a social sense, i.e., payoffs are sustained by a collection of internal organizations of coalitions for which power is distributed over all players in a balanced way. The socially stable core is a subset and therefore a refinement of the core. We show by means of examples that in many cases the socially stable core is a very small subset of the core.

We will state conditions for which the socially stable core is non-empty. In order to derive this result, we formulate a new intersection theorem that generalizes the KKMS intersection theorem. We also discuss the relationship between social stability and the wellknown concept of balancedness for NTU-games, a sufficient condition for non-emptiness of the core. In particular we give an example of a socially structured game that satisfies social stability and therefore has a non-empty core, but whose induced NTU-game does not satisfy balancedness in the general sense of Billera.

KEY WORDS: balancedness, core, non-transferable utility game, social stability.

JEL CLASSIFICATION: C71.

1. INTRODUCTION

In this paper, we generalize cooperative non-transferable utility games (NTU-games) and introduce the concept of a *socially structured game*. A cooperative NTU-game summarizes the results of mutual cooperation by members of a coalition by a set of attainable payoffs. It ignores the fact that coalitions often may organize themselves according to some *internal organization* and that the internal organization may determine both the social strength of its members and the payoff set. In many economic situations of interest, the players within a coalition may choose an internal organization out of a collection of several admissible internal organizations. For instance, in order to run a firm, a group of workers may have multiple possibilities to organize itself, for instance according to different hierarchical graph structures, varying with respect to the number of levels and the span of control. In general, the admissible internal organizations of a group of players depend on the specific application one has in mind. In case of running a firm it may be appropriate to choose the internal organization of a hierarchy. In other cases, networks, communication structures or permutational structures, as studied for transferable utility games by Jackson (2005), Myerson (1977) and Nowak and Radzik (1994), respectively, or ordered structures as studied for NTU-games in van der Laan et al. (1998), might fit better.

Different internal organizations of a coalition may lead to different payoff sets. In this paper, we therefore allow that the set of payoffs attainable to a group of players not only depends on the set of players, but also on the chosen internal organization within the group. A payoff set is associated to any admissible internal organization of any coalition. We further assume that to any admissible internal organization of a coalition a *power vector* is associated, whose components reflect the social strengths of the individual members of the coalition within the internal organization. In the literature several measures have been proposed to determine the strength of an individual in case the internal organization has a (directed) graph structure. One example is to determine the strength of a player as his (out) degree in the graph. However, we will make no attempt to discuss the pros and cons of the various ways a power vector can be determined. The model does not indicate the source of the power. Instead, we will treat the power vectors as exogenously given, and consider its derivation as part of the sociological literature. This treatment parallels the exogenous treatment of preferences in economics or the exogenous distribution of initial endowments in an exchange economy. A similar approach has been followed by Piccione and Rubinstein (2003) who consider an exchange economy in which the exogenous distribution of initial endowments has been replaced by an exogenous given linear ordering of the agents reflecting their strengths: a higher ranked agent is stronger than a lower ranked agent.

In the usual NTU-game approach, the internal organization of a coalition is not specified and for every coalition there is only one admissible payoff set. According to this approach, within a given coalition all members have the same social strength and the payoff sets of the various admissible internal organizations are aggregated to one payoff set. The concept of a socially structured game exploits the availability of different internal organizations by allowing for different payoffs for different internal organizations and by allowing the players to differ in strength by assigning a power vector to any internal organization. Where in the jungle economy of Piccione and Rubinstein (2003) a player can appropriate goods belonging to the player he dominates in the linear order, within our framework a more powerful player within the chosen internal organization is able to select the more preferred payoff vectors from the payoff set at the expense of the less powerful players as far as the less powerful players don't have the possibility to obtain their payoffs within alternative internal organizations in which they are more powerful.

We propose the *socially stable core* as a solution concept for a socially structured game. The socially stable core reflects the idea that the strengths of the players within the internal organization influence the distribution of payoffs among the players. For a payoff vector to be in the socially stable core, there should be neither incentives to deviate from an economic point of view, nor from a social point of view, i.e., a payoff vector should be both economically and socially stable. A payoff vector will be called economically stable if it is an element of the core of the underlying cooperative game, i.e., it is feasible for some admissible organization of the grand coalition and there is no subcoalition that can guarantee all its members a higher payoff by choosing an appropriate internal organization. A payoff vector satisfies social stability when the powers of all players are balanced, meaning that there is a nonnegative weighted combination of the power vectors of all internal organizations that can obtain the payoff vector, which gives each player equal power. The socially stable core is the set of socially stable payoff vectors in the core and is therefore a subset of the core.

We introduce the concept of social stability for a socially structured game and refer to games satisfying this property as socially stable games. A socially structured game is socially stable if every socially stable payoff vector is feasible. It will be shown that a socially stable game has a non-empty socially stable core. To do so, we formulate an intersection theorem on the unit simplex that generalizes the well-known intersection theorem used by Shapley (1973) (see also Herings, 1997; Ichiishi, 1988; van der Laan et al., 1999). Since socially stable games have a non-empty socially stable core, they also have a non-empty core. We show that the socially stable core is typically a small subset of the core. We also demonstrate by an example that social stability of the socially structured game does not imply that the induced cooperative NTU-game is π-balanced in the sense of Billera (1970). Therefore, π-balancedness is a different concept from social stability. Moreover, π-balancedness of an NTU-game does not refine the core, but is a sufficient condition for the non-emptiness of the core. A generalization of π-balancedness that is necessary and sufficient for non-emptiness of the core is given in Predtetchinskii and Herings (2004).

The structure of the paper is as follows. To motivate our approach, in Section 2 we give several examples of games in graph structure, a subset of the general class of socially structured games. In Section 3, we give the general framework of socially structured games and give a formal definition of the solution concept of the socially stable core. In Section 4, the new intersection theorem is presented and we show that a socially stable game has a non-empty socially stable core. In Section 5, we study the relationship between social stability and π-balancedness. Section 6 concludes.

2. SOCIAL STABILITY AND MOTIVATING EXAMPLES

Aumann and Peleg (1960) have introduced cooperative non-transferable utility (NTU) games with a finite number of agents. In a cooperative NTU-game a set of attainable payoff vectors is assigned to each coalition. A payoff vector belongs to the core of a game if it is attainable for the grand coalition and no coalition can improve upon it (see Aumann, 1961).

In this paper, we extend the concept of a cooperative NTU-game by allowing for the possibility that any subset of agents organizes itself according to some internal organization. Each admissible internal organization leads to a non-empty set of attainable payoff vectors. The strengths of the players within a coalition depend on the chosen internal organization and are represented by a power vector. Within a hierarchy a higher ranked player is more powerful than a lower ranked player. In contrast, all players have equal power when decisions are made by consensus. To clarify the idea that payoff distribution should depend on the powers of the players and to motivate the solution concept of socially stable core, in the remaining of this section we consider several examples in which the internal organizations are represented by directed graphs (shortly digraphs).

We denote by $N = \{1, \ldots, n\}$ the set of agents and by $\mathcal{N}$ the collection of non-empty subsets of N. A subset of N is called a coalition and the set N itself is often referred to as the grand coalition. In this section an internal organization of a coalition $S \in \mathcal{N}$ is represented by a digraph $G^S = (S, A)$. Here A is a finite collection of ordered pairs of elements of S. Well-known examples of digraph-structures are the complete graph $A = \{(i, j) | i, j \in S, i \neq j\}$, hierarchies[1] or trees.[2] A digraph $G^S = (S, A)$ is admissible for coalition S if S is able to generate payoffs to its members when being organized according to G^S. We assume that for every coalition $S \in \mathcal{N}$ a (possibly empty) collection of admissible internal organizations is given. This collection is denoted by $\mathcal{G}^S$ and the collection of all admissible digraphs, denoted by $\mathcal{G}$, is obtained by taking the union of $\mathcal{G}^S$ over all non-empty subsets S of N.

The payoff sets are given by a mapping v from $\mathcal{G}$ to the collection of non-empty subsets of $\mathbb{R}^n$ satisfying that for every graph $G^S \in \mathcal{G}^S$, the set $v(G^S) \subset \mathbb{R}^n$ is cylindric with respect to S, i.e., for any two vectors x and y in $\mathbb{R}^n$ with $x_i = y_i$ for all $i \in S$ it holds that $x \in v(G^S)$ if and only if $y \in v(G^S)$. The payoff set $v(G^S)$ associated with graph G^S is the set of attainable payoffs for the players in S when they are organized according to G^S, independent of what the agents outside S do.

We assume that to each digraph $G^S \in \mathcal{G}^S$ an n-dimensional power vector $p(G^S)$ is associated with $p_i(G^S) = 0$ when $i \notin S$ and $p_i(G^S) \geqslant 0$ for $i \in S$. The power vector reflects the social strength of player i within G^S. Although some of the power functions that have been proposed in the literature to measure the strength of the nodes in a digraph will be discussed in Appendix A, we want to stress that throughout the paper it is assumed that in each internal organization of the coalition S, the players in S have exogenously given powers, just as they have exogenously given preferences and initial endowments in an exchange economy, or as the exogenously given linear ordering reflecting the strength of the agents in Piccione and Rubinstein (2003). Within structure G^S the more powerful players are able to take away payoffs of less powerful players in S.

A payoff vector is socially stable if there is collection $\mathcal{B}$ of internal organizations in $\mathcal{G}$ such that the payoff vector is feasible for any internal organization in $\mathcal{B}$ and the powers of all players are balanced, i.e., there are positive weights for the elements in $\mathcal{B}$ such that the weighted sum of the power vectors give equal power to all players. As long as a payoff vector is not socially stable, some players are more powerful than others, and are thereby able to extract payoffs from the players having less strength.

We are interested in payoff vectors that are both socially stable and economically stable, where a payoff vector is said to be economically stable if it belongs to the core. The socially stable core is defined as the set of socially and economically stable payoff vectors. The formal definition is given in Section 3. In the remaining of this section we provide examples to explain and to justify the concept of the socially stable core.

EXAMPLE 2.1 (a principal–agent game). We consider a principal–agent game with player 1 the principal and player 2 the agent. Without cooperating, the players are not able to get any positive payoff, i.e., $v(G^i) = \{x \in \mathbb{R}^2 | x_i \leqslant 0\}$, where G^i denotes the (unique) internal organization of the single player coalition $\{i\}, i=1,2$. Since player 1 is the principal, we describe the internal organization of the two-player coalition $N=\{1,2\}$ by the digraph $G^N=(N,A)$ with $A=\{(1,2)\}$. Taking the score index as the power function (see Appendix A), we obtain $p(G^1)=p(G^N)=(1,0)$ and $p(G^2)=(0,1)$. Suppose that within the two-player coalition the payoffs depend on the level $\theta \in \mathbb{R}$ by which the principal monitors the agent, where the payoff to the principal is equal to 2θ and the payoff to the agent $4(1-\theta)$. Without monitoring ($\theta=0$) the principal gets payoff 0 and the agent 4. These payoffs increase to 2 and decrease to 0, respectively, when the level of monitoring increases from zero to one. We obtain that

$$v(G^N)=\{x \in \mathbb{R}^2 \mid 2x_1+x_2 \leqslant 4\}.$$

The set of economically stable payoff vectors is given by the core

$$C(v)=\{x \in \mathbb{R}^2_+ \mid x_2=4-2x_1\}.$$

For $x \in C(v)$ with $x_2 > 0$ we have that either $x_1 > 0$ and x only belongs to $v(G^N)$, or $x_1 = 0$ and x only belongs to $v(G^N)$ and $v(G^1)$. Since $p(G^N) = p(G^1) = (1, 0)$, in both cases player 1 has more power than player 2 and thus x is not socially stable. Therefore, as long as $x_2 > 0$, player 1 can increase his payoff at the expense of player 2. The unique socially stable vector in the core is the payoff vector $x = (2, 0)$, lying in both $v(G^N)$ and $v(G^2)$, with power vectors (1, 0) and (0, 1), respectively. At this payoff vector the powers of the two players are in balance in the sense that when taking equal weights the weighted sum of the two power vectors associated to the two internal organizations G^N and G^2, both containing $x = (2, 0)$, gives equal power to both players. So, $x = (2, 0)$ is the unique economically and socially stable payoff vector. At this payoff vector player 2 is able to match the power of player 1. In particular, player 2 can leave the coalition and obtain his outside option of zero payoff without cooperating with player 1 and therefore also within the internal organization G^N player 1 cannot further increase his payoff. □

EXAMPLE 2.2 (a linearly ordered game). As a generalization of the previous example, we now consider a firm with a fixed internal organization on N. Within this internal organization agents are linearly ordered in such a way that the internal organization on N is given by the graph $G^N = (N, A)$ where $A = \{(i, i+1) | i = 1, \ldots, n-1\}$. For any subset of N we assume that it can only generate payoffs for its players when they are connected in the graph G^N and stick to the internal organization induced by it. So, when S is connected, so $S = \{j, j+1, \ldots, k\}$ for some j, k with $1 \leqslant j \leqslant k \leqslant n$, then the collection $\mathcal{G}^S$ of internal organizations of S contains only one element, being the graph $G^S = \{(S, A)\}$ with $A = \{(h, h+1) | h = j, \ldots, k-1\}$. When S is not connected, then $\mathcal{G}^S = \emptyset$, i.e., the players of S are not able to generate any payoff. For ease of notation, in the following we denote a coalition of the form $\{j, j+1, \ldots, k\}$ for some j, k with $1 \leqslant j \leqslant k \leqslant n$ by $[j, k]$. We further assume that the power function is such that for any $G^{[j,k]} = ([j, k], A)$ the power vector $p(G^{[j,k]})$ satisfies

$p_h(G^{[j,k]}) > p_{h+1}(G^{[j,k]}), h = j, \ldots, k-1$. This is for instance the case when we take the positional power function (see Appendix A) or when the power of a player is given by the player's number of subordinates.

Since $\mathcal{G}$ only contains graphs of the form $([j,k], A)$ and for every player $h \in N$ such that $h > 1$ we have that $p_{h-1}(G^{[j,k]}) > p_h(G^{[j,k]})$ when $h-1, h \in [j,k]$ and $p_1(G^{[1,k]}) > p_h(G^{[1,k]})$ for any $h, 1 < h \leqslant k$, we obtain that at a payoff vector x the powers of the players can only be balanced when for every player h there exists a coalition of the form $([h,k], A)$ with $k \geqslant h$, such that $x \in v(G^{[h,k]})$. Thus at any socially stable payoff vector x, for every $h \in N$, it must be that x is in a payoff set $v(G^{[h,k]})$ for some $k \geqslant h$, i.e., for every h it must hold that x is in the payoff set of a coalition not containing any superior of h. Obviously, economic stability requires that x is not in the interior of any of these payoff sets, since any payoff vector in the interior of $v(G^{[h,k]})$ is dominated by another attainable payoff vector of coalition $[h,k]$. It follows that at a socially stable payoff vector in the core, for any $h \in N$ there is a coalition $[h,k]$ such that x is on the boundary of the payoff set $v(G^{[h,k]})$.

The set of economically and socially stable payoffs is therefore a subset of the core such that, at any payoff vector in this subset, every player $h \in N$ gets only a share in the payoff he can realize within some coalition $[h,k]$, being a coalition containing only some of his subordinates, but none of his superiors. So, within a firm with a linearly ordered hierarchy, all socially stable profits that a player can realize in cooperation with his superiors is distributed amongst his superiors. All core elements not satisfying this condition cannot be socially stable. The example shows that by using information on the internal organization, the framework of a graph-structured game may provide more precise predictions about the outcome of socially structured economic situations. □

EXAMPLE 2.3 (a small firm three player game). Consider a three player game of a small firm owned by player 1 and with players 2 and 3 as employees. Suppose that on the three player coalition two graph structures are admissible, one reflecting the situation where player 2 is the manager and player 3 the worker and the other one the situation where player 3 is the manager and player 2 the worker. This can be modeled by the two trees on N given by $G_1^N = (N, A_1)$ with $A_1 = \{(1,2),(2,3)\}$ and $G_2^N = (N, A_2)$ with $A_2 = \{(1,3),(3,2)\}$, respectively, i.e., the owner monitors the manager and the manager supervises the worker. Taking the number of subordinates as the power of the players, the respective power vectors are $p(G_1^N) = (2,1,0)$ and $p(G_2^N) = (2,0,1)$. Further assume that the owner needs both other players to make profits, that is the two-player coalitions $\{1,2\}$ and $\{1,3\}$ cannot make any profit. However, the coalition of the two employees has the outside option to leave the firm and to make profit by cooperating together on basis of consensus decisions. So, the (unique) internal organization on $\{2,3\}$ is given by $G^{\{2,3\}} = (\{2,3\}, A)$ with $A = \{(2,3),(3,2)\}$ and power vector $p(G^{\{2,3\}}) = (0,1,1)$. Suppose that the outside options of the single players give them a payoff of at most zero, and that the other payoff sets are given by

$$\begin{aligned} v(G_1^N) &= \{x \in \mathbb{R}^3 | 3x_1 + 2x_2 + x_3 \leqslant 8\}, \\ v(G_2^N) &= \{x \in \mathbb{R}^3 | 3x_1 + x_2 + 2x_3 \leqslant 8\}, \\ v(G^{\{2,3\}}) &= \{x \in \mathbb{R}^3 | x_2 + x_3 \leqslant 2\}. \end{aligned}$$

Taking into account the zero payoff options of the single players, straightforward observations show that the core is given by the non-negative payoff vectors in the set

$$\begin{aligned} C(v) = \{x \in \mathbb{R}^3_+ |\, & \min[3x_1 + 2x_2 + x_3, 3x_1 + x_2 + 2x_3] \\ & = 8, x_2 + x_3 \geqslant 2\}, \end{aligned}$$

being the non-negative payoffs on the upper boundary of the set $v(G_1^N) \cup v(G_2^N)$ satisfying $x_2 + x_3 \geqslant 2$. However, not all core elements are socially stable. In fact, any $x' \in C(v)$ with $x'_2 + x'_3 > 2$ is not socially stable. Clearly, x' does not belong to $v(G^{\{2,3\}})$. Moreover, at least one of the players 2 and 3 gets a positive payoff at x', say player 2, so x' does not belong to $v(G^{\{2,3\}})$. For any $G \in \mathcal{G}$ satisfying $x' \in v(G)$ it holds that

$p_1(G) > p_2(G)$, contradicting that x is socially stable. So, at any socially stable payoff vector in $C(v)$ we must have that $x_2 + x_3 = 2$. Further, also

$$3x_1 + 2x_2 + x_3 = 3x_1 + x_2 + 2x_3,$$

must hold. Suppose not, for instance the left-hand side is greater than the right-hand side, thus $x_2 > x_3$. Since x is in the core and thus the minimum of the right-hand side and the left-hand side is equal to 8, it follows that $3x_1 + 2x_2 + x_3 > 8$, showing that x is not in $v(G_1^N)$. Clearly, $x_2 > 0$, so x does not belong to $v(G^2)$. Moreover, $x_2 + x_3 = 2$ implies that $x_1 > 0$, so x is also not in $v(G_1)$. It follows that x belongs to both $v(G_2^N)$ and $v(G^{\{2,3\}})$, perhaps also to $v(G_3)$, but not to any of the other sets. To balance the powers we need $p(G_2^N)$, since $p_1(G) = 0$ in the other two remaining graphs. However, $p_3(G) > p_2(G)$ for $G = G_2^N$ and $G = G^3$, and $p_3(G) = p_2(G)$ when $G = G^{\{2,3\}}$, contradicting that x is socially stable. Hence, social stability implies $x_2 = x_3 = 1$ and thus $x_1 = 5/3$, and therefore $x = (5/3, 1, 1)$ is the unique element in the core that is socially stable. We want to stress that the unique element in the core that is socially stable belongs to $v(G_1^N)$, $v(G_2^N)$ and $v(G^{\{2,3\}})$. As standard in NTU-games, although the members of the coalition $\{2, 3\}$ can guarantee themselves their payoffs at x, the payoff vector itself can only be guaranteed by the grand coalition. However, different from standard NTU-theory, the grand coalition can still choose one of their internal organizations to realise x, say G_1^N. In this organization player 2 has more power than player 3, which may lead to the question whether x is still stable as soon as G_1^N has been chosen. However, suppose player 2 increases his payoff at the expense of player 3 under the constraint $x_2 + x_3 = 2$. Since in G_1^N the payoffs are bounded by $3x_1 + 2x_2 + x_3$, not only x_3 decreases but also x_1. So, players 1 and 3 will prevent such an increase by enforcing a change in the organization from G_1^N to G_2^N. This possibility of changing the internal organization prevents that players want to deviate from the socially stable outcome. □

3. THE SOCIALLY STABLE CORE

In the previous section, we discussed the concept of socially stable core for socially structured games in case the internal organizations are represented by digraphs. We now formalize the these concepts within a more general setting.

For a given finite set of agents $N=\{1,\ldots,n\}$, we assume that for every coalition $S\in\mathcal{N}$ there exists a collection of social structures according to which the members of the coalition can organize themselves. Within each such internal organization of S certain payoffs can be generated for the members of S. The number of admissible internal organizations for coalition S is assumed to be finite and is denoted by m_S. We allow the number m_S to be zero, in which case there is no way the members of coalition S are able to organize themselves in order to generate payoffs for every member of the coalition. For singleton coalitions $S=\{i\}, i\in N$, we assume that $m_{\{i\}}=1$ and for the grand coalition N we assume that $m_N\geqslant 1$. The collection of admissible internal organizations of coalition S, $S\in\mathcal{N}$, is denoted by $\mathcal{I}^S$. The union over S of all internal organizations for S is denoted by $\mathcal{I}$. To any admissible internal organization of a coalition a *power vector* is associated, measuring the social strength of each player within the corresponding social structure by some exogenously given power function p on the collection of all internal organizations. For an internal organization $I^S\in\mathcal{I}^S$ the number $p_i(I^S)$ denotes the power of agent $i, i\in N$. We assume that any player outside the coalition has zero power, i.e., $p_i(I^S)=0$ for all $i\in N\setminus S$, that the power of every agent within the coalition is non-negative, and that at least one of the agents within the coalition has positive power, i.e., $p_i(I^S)\geqslant 0$ for all $i\in S$ and $\sum_{i\in S}p_i(I^S)>0$. Hence, for every $I^S\in\mathcal{I}^S, \mathcal{S}\in\mathcal{N}$, it holds that $p(I^S)\in\Delta^S$, where $\Delta^S=\{y\in\mathbb{R}^n_+|y_i=0, i\in N\backslash S, \sum_{i\in S}y_i>0\}$.

The payoff sets associated with the internal organizations of the coalitions are represented by a mapping v from $\mathcal{I}$ to the collection of non-empty subsets of $\mathbb{R}^n$. When $x\in v(I^S)$ for some $I^S\in\mathcal{I}^S$, $S\in\mathcal{N}$, this means that if coalition S is internally organized according to social structure I^S, the members of coalition S can attain payoffs $(x_i)_{i\in S}$ for themselves, independent of what the agents outside S are doing. For any $S \subset N$ and $I^S \in \mathcal{I}^S$ the set $v(I^S)$ is assumed to be cylindric with respect to S, i.e., for any two vectors x and y in $\mathbb{R}^n$ with $x_i = y_i$ for all $i \in S$ it holds that $x \in v(I^S)$ if and only if $y \in v(I^S)$. We now have the following definition of a socially structured game (SSG).

DEFINITION 3.1 (Socially structured game). A socially structured game is given by the quadruple $\Gamma = (N, \mathcal{I}, p, v)$ where N is the finite set of players, $\mathcal{I}$ is the finite set of admissible internal organizations, p is a power function assigning to every admissible internal organization a power vector, and v is a mapping assigning to every admissible internal organization a non-empty set of payoff vectors.

In a socially structured game the players have to agree on an internal organization I^N of the grand coalition and a feasible payoff vector $x \in v(I^N)$. Suppose that some $x \in v(I^N)$ does not belong to any other payoff set and that for some pair of players i and j it holds that $p_i(I^N) > p_j(I^N)$. Then we assume that this means that player i has the power to take away payoff from player j. The latter is similar to the approach in Piccione and Rubinstein (2003). However, player i can never force the payoff of player j to a level that j can achieve by himself without any cooperation with other players. Indeed, when the payoff vector also belongs to the payoff set $v(I^{\{j\}})$ of the unique internal organization on the single player coalition $\{j\}$, then since $p_j(I^{\{j\}}) > 0$ and $p_i(I^{\{j\}}) = 0$, there exist positive weights such that the sum of the weighted powers of the players i and j in the internal organizations I^N and $I^{\{j\}}$ are equal to each other. More generally, at a payoff vector $x \in v(I^N)$ player i will not be able to increase his payoff at the expense of player j when x belongs to payoff sets $v(I_\ell), \ell = 1, \dots, k$ such that for positive weights $\lambda_1, \dots, \lambda_k$ the weighted sum of the powers of i and j in the power vectors associated to the k different internal organizations are equal to each other.

The discussion above motivates a solution concept selecting socially stable payoff vectors. A payoff vector is said to be socially stable when there is a collection of internal organizations of coalitions that can attain the payoff vector and at which the powers of all individuals are in balance, so that no individual has the power to take away payoff from other individuals. To give a formal definition of social stability of a payoff vector in an arbitrary SSG $\Gamma=(N,\mathcal{I},p,v)$, we first define the power cone of a payoff vector $x\in\mathbb{R}^n$ as the set.

$$PC(x)=\left\{y\in\mathbb{R}^n|y=\sum_{\{I|x\in v(I)\}}\lambda_I p(I),\lambda_1\geqslant 0 \text{ for all } I\right\}.$$

Notice that the power cone of an arbitrary payoff vector in $\mathbb{R}^n$ is a, possibly empty, cone in $\mathbb{R}^n_+$. The power cone of x is equal to the set of all non-negative linear combinations of power vectors of all internal organizations of coalitions that are able to generate x for its members. A payoff vector is called socially stable if the vector of ones is contained in its power cone.

DEFINITION 3.2 (Socially stable payoff). For a socially structured game $\Gamma=(N,\mathcal{I},p,v)$, a payoff vector $x\in\mathbb{R}^n$ is socially stable if PC(x) contains the n-vector of ones.

Social stability of a payoff vector x thus means that non-negative real numbers or weights can be assigned to the internal organizations that are able to generate x in such a way that the weighted total power of every agent is equal to one and therefore the same for every agent. Therefore no player has the power to seize a higher payoff at the expense of others.

Sometimes it will be useful to define social stability of a collection of internal organizations without reference to a particular payoff vector. Let e denote the n-vector of ones.

DEFINITION 3.3 (Socially stable collection of internal organizations). A collection of internal organizations in $\mathcal{I}$, $\{I_1,\ldots,I_k\}$, is socially stable if the system of equations $\sum_{j=1}^k\lambda_j p(I_j)=e$ has a non-negative solution. A socially stable collection of internal organizations in $\mathcal{I}$ is minimal if no subset of it is socially stable.

A socially stable payoff vector is therefore a payoff vector that can be achieved by every element of a socially stable collection of internal organizations.

The second requirement for a solution concept is that the players agree on an economically stable payoff vector. A payoff vector x is said to be economically stable when it is undominated and can be generated by an internal organization on the grand coalition.

DEFINITION 3.4 (Economically stable payoff). For a socially structured game $\Gamma = (N, \mathcal{I}, p, v)$ a payoff vector x is economically stable if $x \in v(I^N)$ for some $I^N \in \mathcal{I}^N$ and there does not exist an $I \in \mathcal{I}^S$ for some $S \subset N$ and $y \in v(I)$ satisfying $y_i > x_i$ for all $i \in S$.

Economic stability of a payoff vector x thus means that x is feasible for the grand coalition and that there is no internal organization of any coalition that can make all members of that coalition better off than at x. Economic stability thus coincides with the concept of core stability. We therefore refer to the set of all economically stable payoffs of an SSG Γ as the core of Γ.

Socially stable payoff vectors may not be economically stable and reversely. The set of payoff vectors that are both socially and economically stable is called the socially stable core of the game.

DEFINITION 3.5 (Socially stable core). The socially stable core of a socially structured game $\Gamma = (N, \mathcal{I}, p, v)$ consists of the set of socially and economically stable payoff vector of Γ.

A payoff vector x is an element of the socially stable core if there is an internal organization of the whole set of agents that is able to generate x (feasibility), there is no internal organization on a coalition that is able to generate more payoff for its members (undominated), and x can be achieved by a socially stable collection of internal organizations (social stability).

4. NON-EMPTINESS OF THE SOCIALLY STABLE CORE

In this section, we give sufficient conditions for the non-emptiness of the socially stable core of a socially structured game. The most important condition is that the game itself is *socially stable*. A game is called socially stable if every socially stable payoff vector can be sustained by an internal organization on the grand coalition.

DEFINITION 4.1 (Socially stable game). A socially structured game $\Gamma = (N, \mathcal{I}, p, v)$ is socially stable if any socially stable payoff x is feasible for the grand coalition.

Besides social stability of the game the other conditions for non-emptiness of the socially stable core are standard in NTU-theory. All admissible payoff sets should be comprehensive, closed and bounded from above. Comprehensiveness of a payoff set allows the more powerful agents to be able to take away payoffs from the less powerful agents. The degree of non-transferability determines how much the payoffs of the former will increase. Recall that every payoff set of an internal organization of a coalition S is cylindric with respect to S and that $m_{\{i\}} = 1$ for all $i \in N$. In the sequel, the payoff set corresponding to the unique internal organization of the single player coalition $\{i\}$, $i \in N$, is denoted by $v(i)$ and the maximally attainable payoff for agent i is given by the real number α_i.

THEOREM 4.2 (Non-emptiness of the socially stable core). *A socially structured game* $\Gamma = (N, \mathcal{I}, p, v)$ *has a non-empty socially stable core if*

(i) *for every* $S \subset N$*, for every* $I \in \mathcal{I}^S$*, the set* $\{(x_i)_{i \in S}\} | x \in v(I)$ *and* $x_i \geqslant \alpha_i$ *for all* $i \in S\}$ *is bounded*;
(ii) *for every* $I \in \mathcal{I}$*, the set* $v(I)$ *is closed and comprehensive*;
(iii) *the game is socially stable.*

Observe that condition (ii) together with the fact that $v(i)$ is cylindric with respect to $\{i\}$ implies that $v(i) = \{x \in \mathbb{R}^n | x_i \leqslant \alpha_i\}$, for any $i \in N$. We would like to stress that the theorem differs in two ways from the corresponding theorem for NTU-games, saying that a balanced game has a non-empty core. First, the theorem above assures that a socially stable game has not only a non-empty core, but even a non-empty socially stable core. The examples of Section 2 show that usually the set of socially stable core elements is a proper (often even small) subset of the core. In many cases the socially stable core contains only one element. The socially stable core is a refinement of the core, obtained by taking into account that a coalition can be organized according to several internal organizations with associated payoff sets and power vectors measuring the strengths of the players within these organizations. Second, the theorem states the existence of a (socially stable) core vector under the condition of social stability. In the next section, we will point out that social stability does not always imply the existence of a system π such that the corresponding induced standard cooperative NTU-game is π-balanced. The theorem may therefore yield non-emptiness of the core of NTU-games that are not π-balanced for any system π.

In order to prove the theorem we first give an intersection result on the $(n-1)$-dimensional unit simplex Δ defined by

$$\Delta = \left\{ q \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} q_i = 1 \right\}.$$

This intersection result is interesting in itself and generalizes the well-known KKMS intersection theorem (Shapley, 1973). The KKMS lemma says that when Δ is covered by a collection of closed sets $\{C^S \mid S \in \mathcal{N}\}$, then there exists (under conditions similar to (i) and (ii) in the lemma below) a collection of balanced coalitions such that the intersection of the corresponding sets is non-empty. In the lemma below this is generalized to a covering by the collection of closed sets $\{C^I \mid I \in \mathcal{I}\}$ and social balancedness for the power vectors associated to the internal organizations by a power function p on $\mathcal{I}$.

LEMMA 4.3 *Let $\mathcal{I}$ be a finite collection of internal organizations with associated power function $p : \mathcal{I} \to \mathbb{R}^n_+ \backslash \{0\}$, and let $\{C^I \mid I \in \mathcal{I}\}$ be a collection of closed subsets of Δ satisfying*

(i) $\bigcup_{I\in\mathcal{I}} C^I = \Delta$.
(ii) *for every* q *in the boundary of* Δ *it holds that* $S \subset \{i \in N | q_i > 0\}$ *when* $q \in C^I$ *for some* $I \in \mathcal{I}^S$.

Then there exists a socially stable collection $\{I_1, \ldots, I_k\}$ *such that* $\bigcap_{j=1}^k C^{I_j} \neq \emptyset$.

Proof. Without loss of generality we may normalize the power vectors such that $\sum_{i=1}^n p_i(I) = n$ for every $I \in \mathcal{I}$. For $I \in \mathcal{I}$, let us define $c^I = e - p(I)$. Let the set Y^n be defined by

$$Y^n = \mathrm{conv}(\{c^I | I \in \mathcal{I}\}),$$

where $\mathrm{conv}(X)$ denotes the convex hull of a set $X \subset \mathbb{R}^n$. Observe that $\sum_{j=1}^n c_j^I = 0$ for all $I \in \mathcal{I}$ and hence $\sum_{j=1}^n y_j = 0$ for all $y \in Y^n$. Next, define the correspondence $F\colon \Delta \to Y^n$ by

$$F(q) = \mathrm{conv}(\{c^I | q \in C^I, I \in \mathcal{I}\}), \quad q \in \Delta.$$

Since the collection of subsets $\{C^I | I \in \mathcal{I}\}$ is a covering of Δ, the set $F(q)$ is non-empty for all $q \in \Delta$. It is easily verified that, for every $q \in \Delta$, $F(q)$ is convex and compact and that $\bigcup_{q\in\Delta} F(q)$ is bounded. Moreover, since the sets C^I, $I \in \mathcal{I}$, are closed, the mapping $F : \Delta \to Y^n$ is an upper hemi-continuous mapping from the set Δ to the collection of subsets of the set Y^n. Further, both sets Δ and Y^n are non-empty, convex, and compact. Next, let H be the mapping from Y^n to the collection of subsets of Δ defined by

$$H(y) = \{\hat{q} \in \Delta | q^\top y \leqslant \hat{q}^\top y \text{ for every } q \in \Delta\}, \quad y \in Y^n.$$

Clearly, for every $y \in Y^n$ the set $H(y)$ is non-empty, convex, and compact, and H is upper hemi-continuous. Hence, the mapping D from the non-empty, convex, compact set $\Delta \times Y^n$ into the collection of subsets of $\Delta \times Y^n$ defined by $D(q, y) = H(y) \times F(q)$ is upper hemi-continuous and for every $(q, y) \in \Delta \times Y^n$, the set $D(q, y)$ is non-empty, convex, and compact. According to Kakutani's fixed point theorem, the mapping D has a fixed point on $\Delta \times Y^n$, i.e., there exist $q^* \in \Delta$ and $y^* \in Y^n$ satisfying $y^* \in F(q^*)$ and $q^* \in H(y^*)$.

Let $\alpha^* = q^{*\top} y^*$. From $q^* \in H(y^*)$ it follows that $q^\top y^* \leqslant \alpha^*$ for every $q \in \Delta$. By taking $q = e(i)$, where $e(i) \in \Delta$ denotes the i-th unit vector, we obtain that $y_i^* \leqslant \alpha^*$, $i = 1, \dots, n$. Hence,

$$\begin{array}{ll} y_i^* = \alpha^*, & \text{if } q_i^* > 0, \\ y_i^* \leqslant \alpha^*, & \text{if } q_i^* = 0. \end{array} \tag{1}$$

Since $\sum_{i=1}^n y_i^* = 0$, we obtain also that $\alpha^* \geqslant 0$.

On the other hand, $y^* \in F(q^*)$ implies that there exist non-negative numbers $\lambda_1^*, \dots, \lambda_k^*$ satisfying $\sum_{j=1}^k \lambda_j^* = 1$ and $y^* = \sum_{j=1}^k \lambda^* c^{I_j}$ for a collection $\{I_1, \dots, I_k\}$ of k different internal organizations in $\mathcal{I}$ such that $q^* \in C^{I_j}$ for every j, $j = 1, \dots, k$. Without loss of generality we assume that $\lambda_j^* > 0$ for every $j = 1, \dots, k$. Let S_j be the set of agents on which I_j is an internal organization, i.e. $I_j \in \mathcal{I}^{S_j}$ for $j = 1, \dots, k$. By condition (ii) we have that $q_i^* = 0$ implies $i \notin S^j$ for every $j = 1, \dots, k$, and thus $c_i^{I_j} = 1$. Hence,

$$y_i^* = \sum_{j=1}^k \lambda_j^* ci^{I_j} = 1 > 0, \quad \text{if } q_i^* = 0. \tag{2}$$

Suppose there exists an index $i \in N$ such that $q_i^* = 0$. Then it follows from the equations (1) and (2) that $y_i^* > 0$ for all $i \in N$, which contradicts $\sum_{i=1}^n y_i^* = 0$. Consequently, for all $i \in N$, we have that $q_i^* > 0$ and thus $y_i^* = \alpha^*$. Together with $\sum_{i=1}^n y_i^* = 0$ this proves that $y^* = 0^n$. Hence

$$\sum_{j=1}^k \lambda_j^* p(I_j) = e - \sum_{j=1}^k \lambda_j^* c^{I_j} = e - y^* = e$$

and thus the collection $\{I_1, \dots, I_k\}$ is socially stable. Since $q^* \in \bigcap_{j=1}^k C^{I_j}$, this completes the proof. $\square$

The proof of Theorem 4.2 follows by applying Lemma 4.3.

Proof of Theorem 4.2. Without loss of generality we assume that $\alpha_i \leqslant 0$ for all $i \in N$. To apply Lemma 4.3, we define a collection $\{C^I | I \in \mathcal{I}\}$ satisfying the conditions of the lemma and show that an intersection point of a collection of socially stable sets induces an element in the socially stable core of the game. For given $M > 0$ and for any $q \in \Delta$, let the number λ_q be given by

$$\lambda_q = \max\{\lambda \in \mathbb{R} \mid -Mq + \lambda e \in \cup_{I \in \mathcal{I}} v(I)\}.$$

Since $0^n \in v(i)$ and because of conditions (i) and (ii) of the theorem, λ_q exists and is positive for every $M > 0$ and for any $q \in \Delta$. Moreover, following Shapley (1973), using condition (i) of the theorem, the number $M > 0$ can be chosen so large that for every $i \in N$ and $q \in \Delta, q_i = 0$ implies that $i \notin S$ for any $S \subset N$ satisfying $-Mq + \lambda_q e \in v(I)$ for some $I \in \mathcal{I}^S$. Now, for $I \in \mathcal{I}$, define

$$C^I = \{q \in \Delta \mid -Mq + \lambda_q e \in v(I)\}.$$

Since every $v(I), I \in \mathcal{I}$, is closed and comprehensive, the collection of sets $\{C^I | I \in \mathcal{I}\}$ is a family of closed sets covering Δ and satisfies also Condition (ii) of Lemma 4.3. Hence, there is a socially stable collection $\{I_1, \ldots, I_k\}$ of internal organizations in $\mathcal{I}$ such that $\bigcap_{j=1}^k C^{I_j} \neq \emptyset$. Let q^* be a point in this intersection, so $q^* \in C^{I_j}$ for $j = 1, \ldots, k$. Then the point $x^* = -Mq^* + \lambda_{q^*} e$ belongs to $\bigcap_{j=1}^k v(I_j)$, i.e., x^* is a socially stable payoff vector supported by the socially stable collection $\{I_1, \ldots, I_k\}$. Since the game is socially stable we have that $x^* \in v(I^*)$ for some $I^* \in \mathcal{I}^N$, i.e. x^* is feasible. To prove economic stability, suppose there exist an internal organization $I \in \mathcal{I}^S$ for some $S \subset N$ and a payoff vector $y \in v(I)$ such that $y_i > x_i^*$ for all $i \in S$. Since $v(I)$ is comprehensive and cylindric with respect to S, there is a $\mu > 0$ such that $x^* + \mu e \in v(I)$. However, then $-Mq^* + (\lambda_{q^*} + \mu)e \in v(I)$, which contradicts that $-Mq^* + \lambda e \notin v(I)$ for any $\lambda > \lambda_{q^*}$. Hence, x^* cannot be improved upon by any internal organization $I \in \mathcal{I}$, i.e. x^* is also economically stable. This completes the proof. □

Since the socially stable core of a socially structured game is a subset of the core of that game, we have the following corollary.

COROLLARY 4.4. *Let* $(N, \mathcal{I}, p, v)$ *be an SSG satisfying the conditions of Theorem* 4.2. *Then the core of the game is non-empty.*

We conclude this section by considering some characteristics of the socially stable core as a subset of the core. For any element x in the socially stable core of a game Γ feasibility of x implies that $x \in v(I)$ for some $I \in \mathcal{I}^N$. Moreover, there exists a socially stable collection $\mathcal{H} \subset \mathcal{I}$ sustaining x. We noticed already that there does not need to be a socially stable collection of internal organizations on the whole set of agents. Therefore, $\mathcal{H}$ may contain internal organizations of proper subsets of N. Moreover, it might be that x is sustained by several socially stable collections. Now, let $\mathcal{I}(x)$ be the 'supercollection' containing all internal organizations that can achieve x. This collection contains at least one internal organization of the grand coalition N and typically some internal organizations on subsets of N. Economic stability implies that improvements are not possible and therefore x cannot be in the interior of any of these payoff sets. This gives the following corollary.

COROLLARY 4.5 (Boundary property). *For a payoff* x *in the socially stable core of an SSG* Γ *let* $\mathcal{I}(x)$ *be the collection of all internal organizations* I *such that* $x \in v(I)$. *Then* x *is on the boundary of* $v(I)$ *for all* $I \in \mathcal{I}(x)$.

The corollary says that the socially stable core typically selects payoff vectors in the core, which are on the boundary of several payoff sets. In general, an element x of the socially stable core belongs to the (relative) interior of the core if $\mathcal{I}(x)$ contains only internal organizations on N, and x belongs to the boundary of the core if $\mathcal{I}(x)$ contains at least one internal organization on a proper subset of agents (see also Example 2.2).

5. SOCIAL STABILITY AND π-BALANCEDNESS

In this section, we consider the relationship and differences between social stability of socially structured games and

π-balancedness of standard cooperative NTU-games, as introduced in Billera (1970). To define π-balancedness, for any subset $S \in \mathcal{N}$, let $\pi^S \in \mathbb{R}^n_+$ be a vector satisfying $\pi^S_j = 0$ for $j \notin S$ and $\pi^S_i > 0$ for $i \in S$. Then a collection $\{S1, \ldots, S_k\}$ of subsets of N is called π-balanced if there exist positive numbers $\lambda_1, \ldots, \lambda_k$ such that

$$\pi^N = \sum_{j=1}^{k} \lambda_j \pi^{S_j}.$$

Observe that the collection containing only the grand coalition N is balanced. Further, in case for all $S \subset N$ we take $\pi^S_i = 1$ for all $i \in S$, π-balancedness reduces to the well-known balancedness as introduced by Shapley (1973). For generalizations of π-balancedness, which are necessary and sufficient for non-emptiness of the core of an NTU-game, we refer to Predtetchinskii and Herings (2004).

A standard cooperative NTU-game on player set N is defined by a mapping v^c assigning to any $S \in \mathcal{N}$ a non-empty payoff set $v^c(S) \subset \mathbb{R}^n$ which is cylindric with respect to S. For a given π-system $\{\pi^S | S \in \mathcal{N}\}$, an NTU-game (N, v^c) is π-balanced when for any π-balanced collection $\{S1, \ldots, S_k\}$ it holds that

$$\bigcap_{j=1}^{k} v^c(S_j) \subset v^c(N).$$

When every set $v^c(S)$ satisfies the Conditions (i) and (ii) of Theorem 4.2, it is well-known that the NTU-game has a non-empty core when there exists a π-system for which the game is π-balanced. Any π-balanced coalitional game (N, v^c) yields a socially stable game $(N, \mathcal{I}, p, v)$ with, for all $S \subset N$, $m_S = 1$, $v(I^S_1) = v^c(S)$, and $p_i(I^S_1) = \pi^S_i / \pi^N_i, i \in N$. Since $\{N\}$ is π-balanced, $\{N\}$ is also socially stable and thus we have that for this socially structured game obtained from the coalitional game (N, v^c), the socially stable core and the core coincide.

We now reduce a given socially structured game $(N, \mathcal{I}, p, v)$ to a standard cooperative NTU-game (N, v^c) by defining the payoff set mapping function v^c on $\mathcal{N}$ by

$$v^c(S) = \bigcup_{I \in \mathcal{I}^S} v(I), \quad \emptyset \neq S \subset N,$$

i.e., the induced NTU-game payoff set $v^c(S)$ of coalition $S \in \mathcal{N}$ is defined to be the union of all payoff sets assigned to the admissible internal organizations on the coalition S of players. It is straightforward that the core of this induced NTU-game (N, v^c) coincides with the core of the socially structured game $(N, \mathcal{I}, p, v)$. In the previous section we have seen that the socially structure game has a non-empty socially stable core, and thus a non-empty core, when the game is socially stable. Hence, it follows immediately that the induced NTU-game has a non-empty core when the underlying socially structured game is socially stable. The next example shows that social stability of the socially structured game does not necessarily imply that the reduced NTU-game satisfies π-balancedness for some system π.

EXAMPLE 5.1 Let $(N, \mathcal{I}, p, v)$ be a socially structured game with $N = \{1, 2, 3\}, m_{\{1,2\}} = 2, m_{\{1,3\}} = m_{\{2,3\}} = 0$ and $m_S = 1$ for all other $S \subset N$. The mapping v is given by

$$\begin{aligned}
&v(i) = \{x \in \mathbb{R}^3 | x_i \leqslant 0\}, \quad i = 1, 2, 3,\\
&v(I_1^{\{1,2\}}) = \{x \in \mathbb{R}^3 | 2x_1 + x_2 \leqslant 3\},\\
&v(I_2^{\{1,2\}}) = \{x \in \mathbb{R}^3 | x_1 + 2x_2 \leqslant 3\}
\end{aligned}$$

and

$$v(I^N) = v(3) \cap v(I_1^{\{1,2\}}) \cap v(I_2^{\{1,2\}}).$$

The power function p is given by $p(I^i) = e(i)$, where I^i denotes the unique internal organization on the singleton coalition $\{i\}, i = 1, 2, 3, p(I_1^{\{1,2\}}) = (2, 1, 0)^\top, p(I_2^{\{1,2\}}) = (1, 2, 0)^\top$, and $p(I^N) = (1, 1, 1)^\top$. This socially structured game is socially stable. To show this, it should be observed that there are only five minimal socially stable collections, $\{I^1, I^2, I^3\}, \{I_1^{\{1,2\}}, I_2^{\{1,2\}}, I^3\}$, $\{I_1^{\{1,2\}}, I^2, I^3\}, \{I_2^{\{1,2\}}, I^1, I^3\}$, and $\{IN\}$. For each of these collections we have that the intersection of the payoff sets of the members of the collection is a subset of $v(I^N)$. For instance,

$$v(I_1^{\{1,2\}}) \cap v(2) \cap v(3) \subset \left\{ x \in \mathbb{R}^3 | x_1 \leqslant 1\frac{1}{2}, x_2 \leqslant 0, x_3 \leqslant 0 \right\} \subset v(I^N).$$

Because the game is socially stable, the socially stable core is non-empty. In fact, the payoff vector $(1,1,0)^\top$ is the unique element in the socially stable core and is also the unique core element. This payoff vector belongs to $v(I^N)$ and no coalition can improve upon it, so it is in the core. Further, there are no other core elements, since the agents 1 and 2 can improve upon each other point in $v(I^N)$ through one or both of their internal organizations $I_1^{\{1,2\}}$ and $I_2^{\{1,2\}}$. Finally, this payoff vector is sustained through the socially stable collection $\{I_1^{\{1,2\}}, I_2^{\{1,2\}}, I^3\}$.

We now consider the induced NTU-game. The payoff set mapping v^c of this game is given by

$$\begin{aligned} &v^c(i) = v(i), \quad i = 1, 2, 3, \\ &vc(\{1,2\}) = v(I_1^{\{1,2\}}) \cup v(I_2^{\{1,2\}}), \\ &v^c(N) = v(I^N) \end{aligned}$$

and $v^c(S) = \emptyset$ for all other S. Of course, again the payoff vector $(1,1,0)^\top$ is the unique element in the core of this reduced game and thus the core is non-empty. However, there does not exist a π-system for which the game is π-balanced. To show this, first let $\{\pi^S | S \subset N\}$ be a π-system such that $\pi_1^{\{1,2\}} = \pi_2^{\{1,2\}}$. Then the collection $\{\{1,2\},\{3\}\}$ is π-balanced. However, $v^c(\{1,2\}) \cap v^c(3)$ is not contained in $v^c(N)$, for instance $x = (1/2, 2, 0)^\top$ is in $v^c(\{1,2\}) \cap v^c(3)$ but not in $v^c(N)$. Hence, the game is not π-balanced for any π-system with $\pi_1^{\{1,2\}} = \pi_2^{\{1,2\}}$. Next, suppose the latter equality does not hold. In that case we assume without loss of generality that $\pi_1^{\{1,2\}} < \pi_2^{\{1,2\}}$. Then the collection $\{\{1,2\},\{1\},\{3\}\}$ is π-balanced. However, the payoff vector $x = (0,3,0)^\top$ is in $v^c(\{1,2\}) \cap v^c(1) \cap v^c(3)$ but not in $v^c(N)$ and again the game is not π-balanced. Hence, there does not exist a π-system for which the induced NTU-game is π-balanced, so that the non-emptiness of the core cannot be concluded from the π-balancedness condition. This concludes the example. $\square$

6. CONCLUDING REMARKS

In this paper, we have introduced socially structured games. The concept of socially structured games extends the standard cooperative NTU-game setting by taking into account that the players in a coalition may organize themselves according to several internal organizations. To any internal organization a set of payoff vectors that the members of the coalition can achieve is associated. The strengths of the players within a coalition depends on the internal organization and are given by an exogenously given power vector. More power means that payoffs can be taken away from less powerful players. We would like to stress once more that we allow for more than one admissible internal organization on the same coalition. Since the class of socially structured games is sufficiently rich to encompass many economic phenomena of interest, we believe it to be of high potential for further research.

The members of the grand coalition have to agree on a feasible payoff vector. We take the point of view that distribution of the payoffs depends on the strengths of the players within the different internal organizations. This is captured by the solution concept of the socially stable core. This solution is a refinement of the core obtained by the additional requirement that a payoff distribution should be sustained by a collection of internal organizations such that all players have equal power. As long as powers are not balanced, a more powerful player is able to seize a higher payoff at the expense of less powerful players.

We motivated the paper by considering a special class of socially structured games in more detail, the so-called digraph games. Similarly we may consider games in which the social structure is given by an undirected graph. Such a graph may represent a communication structure or network. In such a case a possible interpretation of the strengths of the players is their position within the network, measured by a so-called centrality measure. In our approach a set of payoffs is associated to any admissible network and the allocation of payoffs depends on the exogenously given powers of the players within the various admissible networks. This approach differs from the network literature as initiated by Jackson and Wolinsky (1996) (see also Jackson, 2005). In that literature players can form any network on the set of all players by adding or deleting links and any network yields a so-called network value. Within a given network this value is distributed among the players according to some exogenously given allocation rule. By imposing some stability requirement with respect to adding or deleting links, a first question is which network will be formed by the players under the given allocation rule. Since in that approach the allocation rule plays a central role, a further question is which axiomatic properties the allocation rule should satisfy. In Jackson (2005) a family of allocation rules is proposed that incorporate information about alternative networks when allocating the value of a given network. Our research can be seen as advocating a (not necessarily unique) allocation rule in which the powers of the players within the different networks are taken into account.

APPENDIX A

In the literature several power functions have been proposed to measure the power of the nodes within a digraph. To give some examples of such power functions, we define the sets of predecessors, successors and subordinates of a node $i \in S$ of a graph $G^S = (S, A)$ by

$$
\begin{aligned}
&P^i(G^S) = \{j \in S | (j, i) \in A\},\\
&D^i(G^S) = \{j \in S | (i, j) \in A\} \text{ and}\\
&\widehat{D}^i(G^S) = \{j \in S | \text{ there is a path from } i \text{ to } j\},
\end{aligned}
$$

respectively, i.e., $P^i(G)$ is the set of all predecessors of node i in G, $D^i(G)$ is the set of all successors of node i in G and $\widehat{D}^i(G)$ is the set of subordinates of player i in G, so for any player j in $\widehat{D}^i(G)$ there is a directed path of subsequent arcs from i to j. A well-known power function to measure the power of a node in a graph is the *score index*, see for instance Behzad et al., 1979; or Rubinstein, 1980. According to the score index, the power of a node $i \in S$ in the graph $G^S = (S, A)$ is equal to the number of elements in the set $D^i(G^S)$, i.e., the number of successors of i in G^S. For hierarchies a straightforward alternative is to take as the power of a player the number of the elements in the subordinate set $\widehat{D}^i(G^S)$, i.e., to take the score index of the transitive closure of the graph.[3] Another power function has been introduced by van den Brink (1994) (see also van den Brink and Gilles, 2000), according to which the power of a node $i \in S$ in a graph $G^S = (S, A)$ is given by $\sum_{j \in D^i(G^S)} |P^j(G^S)|^{-1}$. The interpretation of this *dominance index* is as follows. Initially, each node gets one point. This point is equally distributed amongst all its predecessors, i.e., amongst all the nodes by which a node is dominated. The power of a node is then the sum of all its shares in the points of its successors.

The third power function we discuss is the *positional power index*, introduced in Herings et al. While the score index of a node only depends on its number of successors and the dominance index of a node on the number of predecessors of each of its successors, the positional index of a node depends both on the number of its successors and on how powerful its successors are. More precisely, for a digraph $G^S = (S, A)$ the positional indices of the players in S is given by the solution to the system of linear equations

$$x_i = \sum_{j \in D^i(G^S)} (1 + bx_j), \quad i \in S, \tag{3}$$

for some given non-negative number b. It means that any node gets for each successor a power of 1 plus a multiple b of the power of that successor. The positional index of a node is higher if it its successors are more powerful. As shown in Herings et al., if A is non-empty, then the system (3) has a unique non-negative non-zero solution for any b, $0 \leqslant b \leqslant 1/n$. In case the graph is a tree it can be shown that the system (3) has a solution for any $b \geqslant 0$. For the choice $b \geqslant 1$ it follows straightforwardly that a node has always more power than any of its successors, so a player gets more power when it is higher in the tree. For $b = 1$ the number of subordinates in a tree is obtained.

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NOTES

1. A digraph is a hierarchy if it does not contain a cycle.
2. A digraph is a tree if it is a hierarchy and there is a unique node, called the root, such that there is a unique path from the root to any other node.
3. The transitive closure of a digraph (S, A) is the graph $(S, \hat{A})$ defined by $(i, j) \in \hat{A}$ iff (S, A) contains a path from i to j.

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